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Abstract

The inverse Laplace transform of the image function $p^{\alpha-1}\exp(-a^{\frac{1}{m}}\,p^{\frac{1}{m}})$, which arises in electromagnetic diffraction theory, is obtained and the asymptotic behavior of the original function for values of the independent variable near 0 is then determined.

1. Introduction

In electromagnetic problems one often has occasion to study pulse fields, that is, fields created by sources which have a Heaviside unit function behavior or a delta-function behavior in the time variable as well as fields created by the same sources and having a harmonic time behavior. The latter fields also prove to be harmonic in time (apart from a transient) and these time harmonic fields are the Laplace transforms of the corresponding pulse fields[7]. It is also true that if one expands the pulse solution of Maxwell's equations (or the scalar wave equation) in powers of $(t-t_o)$ where t_o is a singularity of the pulse solution and if one obtains an asymptotic series representation of the spatial behavior of the corresponding time harmonic field in powers of $1/\omega$ where ω is the circular frequency of the source, then the individual terms of the asymptotic series are essentially Laplace transforms of the corresponding terms in the series of powers of $t-t_o$ (see eq. (4.8) of [8]).

To extend the knowledge of the relationship between time harmonic fields and pulse fields and between the asymptotic series described above and the corresponding series of powers to fields diffracted by smooth bodies, it now appears desirable to utilize some Laplace transforms which have not been established. The present paper is concerned with determining the original function whose Laplace transform is

$$(1) \qquad p^{\alpha-1} \exp\left[-a^{1/m}\, p^{1/m}\right],$$

where p, as usual, is complex, $R(p) > 0$, $R(\alpha) > 0$, $R(a) > 0$ and $m = 2,3,4,\ldots$. It is expected that the result will be useful in electromagnetic diffraction problems but the present paper will be devoted to the purely mathematical problem of determining the original function and also the asymptotic behavior of the original function for t near 0.

When m = 2 the function (1) is, of course,

(2) $$p^{\alpha-1} \exp\left[-a^{1/2} p^{1/2}\right]$$

and the original function is known (see [1], p. 246, eq. 9). It is in fact the function

(3) $$2^{\frac{1}{2}-\alpha} \pi^{-\frac{1}{2}} t^{-\alpha} e^{-\frac{a}{8t}} D_{2\alpha-1}\left(2^{-\frac{1}{2}} a^{\frac{1}{2}} t^{-\frac{1}{2}}\right)$$

where $R(a) > 0$, $R(\alpha) > 0$ and $D_\nu(z)$ is the parabolic cylinder function (see [2], p. 392)

$$D_\nu(z) = 2^{\frac{\nu}{2}+\frac{1}{4}} z^{-\frac{1}{2}} W_{\frac{\nu}{2}+\frac{1}{4}, \pm\frac{1}{4}}\left(\frac{z^2}{2}\right)$$

(4) $$= \frac{2^{-\nu-1} z^\nu}{\pi^{1/2}\Gamma(-\nu)} e^{-z^2/4} E\left(-\frac{\nu}{2}, \frac{1}{2}-\frac{\nu}{2} :: \frac{z^2}{2}\right) .$$

The function appearing on the right side of this last equation is MacRobert's E-function whose definition and properties are to be found in the literature (see [3], p. 348 and 352 or [6], pp. 203-206) and about which more will be said below. Hence the major result of this paper is a generalization of the Laplace transform relationship between the functions (3) and (2) to the case where the exponents in the exponential part of (2) are $1/m$. The result will be stated and derived in Section 2. After deriving a subsidiary formula in Section 2 we show in Section 4 that the general result of Section 2 reduces to the special case when $m = 2$. In Section 5 we derive the asymptotic behavior as t approaches 0 of the inverse Laplace transform of the function (1).

2. Derivation of the original function

We present first for convenient reference several facts about the E-function and other relationships which will be utilized later. We begin with some facts about the generalized hypergeometric function.

The symbols $(k;0)$ and $k;n)$, where k may be real or complex, are defined as

$$(k;0) = 1$$

$$(k;n) = k(k+1)(k+2)\ldots(k+n-1), \qquad n = 1,2,3,\ldots \quad .$$

Then the generalized hypergeometric function $F(p;\alpha_r:q;\rho_s:z)$ is defined as

$$F(p;\alpha_r:q;\rho_s:z) = \sum_{n=o}^{\infty} \frac{(\alpha_1;n)(\alpha_2;n)\ldots(\alpha_p;n)}{n!(\rho_1;n)(\rho_2;n)\ldots(\rho_q;n)} z^n$$

wherein it is assumed that none of the ρ's is zero or a negative integer. An alternative notation for this F-function is

$$F\begin{pmatrix} \alpha_1,\alpha_2,\ldots,\alpha_p;z \\ \rho_1,\rho_2,\ldots,\rho_q \end{pmatrix} .$$

When $p = 1$ and $q = 1$, the generalized hypergeometric function $F(1;\alpha:1;\rho:z)$ is often denoted as $F(\alpha;\rho;z)$ or as ${}_1F_1(\alpha;\rho;z)$.

If $p \leq q$ then the E-function is defined as

$$E(p;\alpha_r:q;\rho_s:z) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\ldots\Gamma(\alpha_p)}{\Gamma(\rho_1)\Gamma(\rho_2)\ldots\Gamma(\rho_q)} F(p;\alpha_r:q;\rho_s: -\frac{1}{z})$$

where $z \neq 0$. If $p \geq q+1$, and this is the case which will be of interest here, and when $|\text{amp } z| < \pi$, then the E-function (see p. 353) can be shown to be [3] *

$$E(p;\alpha_r:q;\rho_s:z) = \sum_{r=1}^{p} \prod_{s=1}^{p}{}' \Gamma(\alpha_s-\alpha_r) \left\{ \prod_{t=1}^{q} \Gamma(\rho_t-\alpha_r) \right\}^{-1} \Gamma(\alpha_r) \cdot \tag{5}$$

$$z^{\alpha_r} F\begin{pmatrix} \alpha_r,\alpha_r-\rho_1+1,\ldots\alpha_r-\rho_q+1;(-1)^{p-q}z \\ \alpha_r-\alpha_1+1,\ldots * \ldots, \alpha_r-\alpha_p+1 \end{pmatrix}$$

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* The definition of the E-function for this case is given in the Appendix where it is utilized.

where the dash in the product sign signifies that the factor for which $s = r$ is omitted and the asterisk in the F function means that the parameter $\alpha_r - \alpha_r + 1$ is omitted, so that there are only p-1 parameters in the lower set. When there are no ρ_s's in the E-function, that is, when $q = 0$, then equation (5) holds under the conditions that the factor in the braces is omitted and the parameters $\alpha_r - \rho_1 + 1, \ldots, \alpha_r - \rho_q + 1$ are omitted in the F-function.

To familiarize ourselves with the E-function the following relations may be worth noting though they will not be used. From the definition of the E-function for the case $p \leq q$ and from equation (5) it is clear that the E-function is immediately related to the generalized hypergeometric function and reduces to simple expressions in the ordinary or Gauss hypergeometric function when $p = 2$ and $q = 1$. For $p = 1$ and $q = 1$ it is also evident from the definition that the F-function reduces to the confluent hypergeometric function or Kummer's function. The case $p = 2$, $q = 0$ yields the following relation (see [3], p. 351, eq. (14))

$$\cos n\pi E(\tfrac{1}{2} + n, \tfrac{1}{2} - n :: 2z) = \sqrt{2\pi z}\; e^{z}\, K_n(z)$$

where $K_n(z)$ is the modified Bessel function of the second kind. It is also true that

$$E(\tfrac{1}{2} - k + m, \tfrac{1}{2} - k - m :: z) = \Gamma(\tfrac{1}{2} - k + m)\Gamma(\tfrac{1}{2} - k - m) z^{-k} e^{\frac{1}{2} z}\, W_{k,m}(z)$$

where $W_{k,m}(z)$ is the Whittaker function (see [3], p. 351, eq. (15)). Finally it is immediate from the definition of the E-function that for $p = 0$ and $q = 0$ the F-function is just $e^{-1/z}$. More complicated parameters in the E-function lead to the equivalence of the E-function with products of Hankel functions, with Lommel functions, and with products of Whittaker functions (see [3], p. 395).

We shall utilize the relationship

$$(6) \qquad \int_0^{\infty} e^{-\lambda} \lambda^{\alpha_{p+1} - 1} E(p;\alpha_r : q;\rho_s : z/\lambda)\, d\lambda = E(p+1;\alpha_r : q;\rho_s : z)$$

where $R(\alpha_{p+1}) > 0$. This equation will be proven in Appendix I.

Other equations we shall use are

$$(7)\qquad \Gamma(z)\Gamma(z+\tfrac{1}{m})\ \ldots\ \Gamma(z+\tfrac{m-1}{m}) = (2\pi)^{\frac{m}{2}-\frac{1}{2}}\, m^{\frac{1}{2}-mz}\,\Gamma(mz)$$

which is the multiplication equation of Gauss and Legendre (see [6], p. 4, eq. (11));

$$(8)\qquad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}$$

(see [6], p. 3, eq. (6)) ;

$$(9)\qquad {}_1F_1(\alpha;\rho;z) = e^z\, {}_1F_1(\rho-\alpha;\rho;-z)$$

(see [3], p. 346, eq. (5)); and the common trigonometric identity

$$(10)\qquad \sin\frac{\pi}{m}\sin\frac{2\pi}{m}\ \ldots\ \sin\frac{m-1}{m}\pi = 2^{1-m}m.$$

We shall be interested in the E-function when there are no ρ_s's, that is, when $q = 0$ and when the p parameters α_r are the m parameters α, $\alpha+\frac{1}{m}$, $\alpha+\frac{2}{m}$, ..., $\alpha+\frac{m-1}{m}$. Then we shall use the accepted notation

$$E(\alpha,\alpha+\tfrac{1}{m},\ \alpha+\tfrac{2}{m},\ \ldots,\ \alpha+\tfrac{m-1}{m}::z)\ .$$

From (5) with the conventions agreed upon in connection with (5) we may say that

$$(11)\qquad E(\alpha,\alpha+\tfrac{1}{m},\ldots,\alpha+\tfrac{m-1}{m}::z) = \sum_{r=0}^{m-1}\prod_{s=1}^{\nu}\Gamma(-\tfrac{s}{m})\prod_{t=1}^{m-\nu-1}\Gamma(\tfrac{t}{m})\Gamma(\alpha+\tfrac{r}{m})\; z^{\alpha+\frac{r}{m}}\, F\begin{pmatrix}\alpha+\frac{r}{m}\ ;\ z\\ 1+\frac{r}{m},\ 1+\frac{r-1}{m},\ldots,1+\frac{1}{m},1-\frac{1}{m},1-\frac{2}{m},\ldots,1-\frac{m-1-\nu}{m}\end{pmatrix}$$

wherein we have split the first product on the right side of (5) into two separate products. The F-function on the right side of (11) is sometimes also accompanied by subscripts thus: ${}_1F_{m-1}$.

We shall now establish the following Laplace transform theorem in which

the original function is the function of t and p is the Laplace transform parameter whose real part is greater than 0:

$$\int_0^\infty e^{-pt} \sum_{i,-i} \frac{1}{i} E(\alpha,\alpha+\frac{1}{m},\alpha+\frac{2}{m},\ldots,\alpha+\frac{m-1}{m} :: \frac{ae^{i\pi}}{m^m t})dt = \tag{12}$$

$$(2\pi)^{\frac{1}{2}+\frac{m}{2}} m^{-\frac{1}{2}-m\alpha} a^\alpha p^{\alpha-1} \exp\left[-a^{1/m} p^{1/m}\right]$$

where $R(\alpha) > 0$, $R(a) > 0$, $m = 2,3,4,\ldots$, and $\sum_{i,-i}$ means that in the expression following it i is to be replaced by -i and the two expressions are to be added.

To prove equation (12) we shall transform the left side into the right. We consider therefore the left side. We substitute $t = \lambda/p$ and apply (6) in which we let the value of $\alpha_{p+1} = 1$. This yields

$$\int_0^\infty e^{-pt} \sum_{i,-i} \frac{1}{i} E(\alpha,\alpha+\frac{1}{m},\alpha+\frac{2}{m},\ldots,\alpha+\frac{m-1}{m} :: \frac{ae^{i\pi}}{m^m t})dt = \tag{13}$$

$$\frac{1}{p}\sum_{i,-i} \frac{1}{i} E(\alpha,\alpha+\frac{1}{m},\alpha+\frac{2}{m},\ldots,\alpha+\frac{m-1}{m},1 :: \frac{ape^{i\pi}}{m^m}).$$

We now express each E function on the right side in terms of the value given by (11) and combine the two resulting expressions by factoring out common terms. The terms which result from $z^{\alpha+r/m}$ in (11) are transformed by means of (8). We also use the definition of the generalized hypergeometric function to cancel factors in numerator and denominator. Hence the right side of the preceding equation becomes

$$(\frac{2\pi}{p})(\frac{ap}{m^m})^\alpha \sum_{r=0}^{m-1} \left[(\frac{ap}{m^m})^{\frac{r}{m}} \Gamma(-\frac{r}{m})\Gamma(-\frac{r-1}{m}) \cdots \Gamma(-\frac{1}{m})\Gamma(\frac{1}{m})\Gamma(\frac{2}{m})\cdots\Gamma(\frac{m-1-r}{m})\right. \tag{13a}$$

$$\left. \cdot\, {}_0F_{m-1}(;1+\frac{r}{m},1+\frac{r-1}{m},\ldots,1+\frac{1}{m},1-\frac{1}{m},1-\frac{2}{m},\ldots,1-\frac{m-1-r}{m};\frac{(-1)^m ap}{m^m})\right]$$

where the notation in ${}_0F_{m-1}$ means

$$F\left(\begin{matrix} & \dfrac{(-1)^{m} ap}{m^m} \\ 1+\frac{r}{m},1+\frac{r-1}{m},\ldots,1+\frac{1}{m},1-\frac{1}{m},1-\frac{2}{m},\ldots,1-\frac{m-1-r}{m} & \end{matrix}\right).$$

Now apart from the power $(ap)^{\alpha}$ before the summation in (13a) the expression contains powers of $(ap)^{s}$ where $s = 0,1,2,\ldots$ in the generalized hypergeometric function each multiplied by $\sum_{r=0}^{m-1} (ap)^{r/m}$. Let us however concentrate for the moment on a particular value of s and on a particular value of r, the former in the range $0,1,2,\ldots$ and the latter in the range from 0 to m-1. By using the series meaning of ${}_0F_{m-1}$ we find that the coefficient of $(ap)^{r/m+s}$ in (13a) equals

$$\frac{2\pi}{p} m^{-m\alpha-r-ms}(ap)^{\alpha}(-1)^{ms} \cdot$$

(14)
$$\left[\frac{\left\{\Gamma(-\frac{r}{m})\Gamma(1+\frac{r}{m})\cdots\Gamma(-\frac{1}{m})\Gamma(1+\frac{1}{m})\right\}\left\{\Gamma(\frac{1}{m})\Gamma(1-\frac{1}{m})\cdots\Gamma(\frac{m-r-1}{m})\Gamma(1-\frac{m-r-1}{m})\right\}}{s!\left\{\Gamma(1+\frac{r}{m}+s)\cdots\Gamma(1+\frac{1}{m}+s)\right\}\left\{\Gamma(1-\frac{1}{m}+s)\cdots\Gamma(1-\frac{m-1-r}{m}+s)\right\}}\right].$$

We now use (8) on the numerator of this expression to obtain

$$\frac{2\pi}{p} m^{-m\alpha-r-ms}(ap)^{\alpha}(-1)^{r+ms} \frac{\pi^{m-1}}{\left\{\sin\frac{\pi}{m}\sin\frac{2\pi}{m}\cdots\sin\frac{r\pi}{m}\right\}\left\{\sin\frac{\pi}{m}\sin\frac{2\pi}{m}\cdots\sin\frac{m-r-1}{m}\pi\right\}} \cdot$$

(15)
$$\frac{1}{\Gamma(1+s)\left\{\Gamma(1-\frac{m-1-r}{m}+s)\ldots\Gamma(1-\frac{1}{m}+s)\right\}\left\{\Gamma(1+\frac{1}{m}+s)\ldots\Gamma(1+\frac{r}{m}+s)\right\}} \cdot$$

We now use the simple identity $\sin\frac{r\pi}{m} = \sin(\pi-\frac{r}{m}\pi) = \sin\frac{m-r}{m}\pi$, for $r = 1,2,\ldots,$ m-1 in the first denominator and rewrite the factors in the second denominator with a trivial change in form. Hence (15) becomes

$$\frac{2\pi}{p} m^{-m\alpha-r-ms}(ap)^{\alpha}(-1)^{r+ms} \frac{\pi^{m-1}}{\sin\frac{\pi}{m}\sin\frac{2\pi}{m}\cdots\sin\frac{r\pi}{m}\sin\frac{r+1}{m}\pi\sin\frac{r+2}{m}\pi\ldots\sin\frac{m-1}{m}\pi} \cdot$$

(16)
$$\frac{1}{\Gamma(\frac{1}{m}+\frac{r}{m}+s)\Gamma(\frac{2}{m}+\frac{r}{m}+s)\ldots\Gamma(\frac{1}{m}+\frac{r}{m}+s+\frac{m-1}{m})} \cdot$$

Use of relations (10) and (7) converts (16) to

$$\frac{2\pi}{p} m^{-m\alpha-r-ms}(ap)^{\alpha}(-1)^{r+ms} \frac{\pi^{m-1}}{m2^{1-m}} \frac{1}{m^{\frac{1}{2}-1-r-ms}(2\pi)^{\frac{m}{2}-\frac{1}{2}}\Gamma(1+r+ms)} .$$

Thus the coefficient of $(ap)^{\frac{r}{m}+s}$ in (13a) becomes

$$(17) \qquad (2\pi)^{\frac{1}{2}+\frac{m}{2}} m^{-\frac{1}{2}-m\alpha} a^{\alpha} p^{\alpha-1} \frac{(-1)^{r+ms}}{\lfloor r+ms},$$

which equals

$$(2\pi)^{\frac{1}{2}+\frac{m}{2}} m^{-\frac{1}{2}-m\alpha} a^{\alpha} p^{\alpha-1} \left\{ \text{coef. of} (ap)^{\frac{1}{m}(r+ms)} \text{ in the expansion of } e^{-a^{\frac{1}{m}} p^{\frac{1}{m}}} \right\} .$$

If we now consider the successive powers of $(ap)^{1/m}$ in the series expansion of $e^{-a^{1/m}p^{1/m}}$ we see that we shall get first of all the set of powers from $n = 0$ to $n = m-1$ which gives the powers of $(ap)^{r/m+s}$ for $s = 0$ and r from 0 to $m-1$. The powers from $n = m$ to $n = m+\frac{m-1}{m}$ in the exponential series gives the powers of $(ap)^{\frac{r}{m}+s}$ for $s = 1$ and r from 0 to $m-1$. More generally every power of $(ap)^{\frac{r}{m}+s}$ which is called for in (13a) (apart from the factor $(ap)^{\alpha}$ in front) is contained in the series for $e^{-a^{1/m}p^{1/m}}$ and only those powers are contained in the exponential series. Thus (13) becomes

$$\int_0^{\infty} e^{-pt} \sum_{i,-i} \frac{1}{i} F(\alpha,\alpha+\frac{1}{m},\alpha+\frac{2}{m},\ldots,\alpha+\frac{m-1}{m} :: \frac{ae^{i\pi}}{m^m t})\, dt =$$

$$(18) \qquad (2\pi)^{\frac{1}{2}+\frac{m}{2}} m^{-\frac{1}{2}-m\alpha} a^{\alpha} p^{\alpha-1} \exp(-a^{1/m}p^{1/m}),$$

which is the desired result as stated in (12).

3. Derivation of a subsidiary equation

We wish to show that the general transform (12) or (18) reduces in the case $m = 2$ to the Laplace transform relationship between (3) and (2). For this purpose we shall first prove a subsidiary equation, namely,

$$(19)\quad \frac{2\pi i z^{\alpha+\beta-1} e^{-z}}{\Gamma(1-\alpha)\Gamma(1-\beta)} E(1-\alpha,1-\beta::z) = E(\alpha,\beta::ze^{i\pi}) - E(\alpha,\beta::ze^{-i\pi}),$$

where the notation in the E-functions is explained by (11).

We re-express each of the two E-functions on the right side of (19) by means of equation (11), add similar terms, and obtain

$$E(\alpha,\beta::ze^{i\pi}) - E(\alpha,\beta::ze^{-i\pi})$$

$$(20)\quad = 2\pi i\left[\frac{\Gamma(\beta-\alpha)}{\Gamma(1-\alpha)} z^{\alpha}\ {}_1F_1\binom{\alpha;\ -z}{1-\alpha-\beta} + \frac{\Gamma(\alpha-\beta)}{\Gamma(1-\beta)}\ {}_1F_1\binom{\beta;\ -z}{1+\beta-\alpha}\right].$$

We now use relation (9) to write

$$= \frac{2\pi i z^{\alpha+\beta-1}}{\Gamma(1-\alpha)\Gamma(1-\beta)} e^{-z}\left\{\begin{array}{l}\Gamma(\alpha-\beta)\Gamma(1-\alpha)z^{1-\alpha}\ {}_1F_1\binom{1-\alpha;z}{1+\beta-\alpha} + \\ \quad \Gamma(\beta-\alpha)\Gamma(1-\beta)z^{1-\beta}\ {}_1F_1\binom{1-\beta;z}{1+\alpha-\beta}\end{array}\right\}.$$

By applying (11) we obtain (19) above. Hence the subsidiary equation is established.

4. Derivation of the case $m = 2$

For the case $m = 2$ the general result (12) tells us that the transform of

$$(21)\quad \frac{1}{1} E(\alpha,\alpha+\tfrac{1}{2} :: \frac{ae^{i\pi}}{4t}) - \frac{1}{1} E(\alpha,\alpha+\tfrac{1}{2} :: \frac{ae^{-i\pi}}{4t})$$

is

$$(22)\quad (2\pi)^{\frac{3}{2}}\, 2^{-\frac{1}{2}-2\alpha}\, a^{\alpha}\, p^{\alpha-1} \exp(-a^{\frac{1}{2}} p^{\frac{1}{2}}).$$

We shall show that the transform relationship between expressions (21) and (22) agrees with the known relationship stated in expressions (3) and (2).

We use the equality (19) to transform (21) and obtain

$$\frac{2\pi(\frac{a}{4})^{2\alpha-\frac{1}{2}}t^{\frac{1}{2}-2\alpha}}{\Gamma(1-\alpha)\Gamma(\frac{1}{2}-\alpha)}\quad e^{-a/4t}\ E(1-\alpha,\ \tfrac{1}{2}-\alpha :: \tfrac{a}{4t}).$$

We now use equation (4) wherein we let $\nu = 2\alpha-1$ to obtain from the preceding expression

$$\frac{2\pi(\frac{a}{4})^{2\alpha-\frac{1}{2}}t^{\frac{1}{2}-2\alpha}}{\Gamma(1-\alpha)\Gamma(\frac{1}{2}-\alpha)}\quad e^{-a/4t}\ \frac{\pi^{1/2}\Gamma(1-2\alpha)}{2^{-2\alpha}(\frac{a}{2t})^{\alpha-1/2}}\ e^{a/8t}D_{2\alpha-1}\left(\frac{a^{1/2}}{2^{1/2}t^{1/2}}\right).$$

By means of the duplication formula for the gamma function, namely,

$$\Gamma(2x) = \Gamma(x)\Gamma(x+\tfrac{1}{2})\ \frac{2^{2x-1}}{\sqrt{\pi}}$$

wherein we let $x = \frac{1}{2} - \alpha$ we may simplify the preceding expression to

$$2^{-3\alpha+\frac{3}{2}}\ \pi\ a^{\alpha}t^{-\alpha}\ e^{-a/8t}\ D_{2\alpha-1}\left(\frac{a^{1/2}}{2^{1/2}t^{1/2}}\right).$$

If we now eliminate constant factors in this expression and in (24) we see that the Laplace transform relationship between (21) and (22) is identical with the Laplace transform relationship between (3) and (2).

5. The asymptotic behavior of the original function for t near 0

It is known for a wide class of original functions and their Laplace transforms that the asymptotic behavior of the original function for t near 0 determines the asymptotic behavior of the image function as $|p|$ becomes infinite (see [5], p.137). With a possible extension of this existing theorem in mind we determine here the asymptotic behavior of the original function

(25) $$(2\pi)^{-\frac{1}{2}-\frac{m}{2}}\ m^{\frac{1}{2}+m\alpha}\ a^{-\alpha}\sum_{i,-i}\frac{1}{i}\ E(\alpha,\alpha+\tfrac{1}{m},\ \alpha+\tfrac{2}{m},\ \ldots,\alpha+\tfrac{m-1}{m} :: \frac{ae^{i\pi}}{m^m t}).$$

We utilize a known integral for the E-function called the Barnes integral (see [3], p. 374). For our case where the E-function contains no ρ_s's and the

p parameters α_r are α, $\alpha+\frac{1}{m}$, $\alpha+\frac{2}{m}$, ..., $\alpha+\frac{m-1}{m}$, then

$$E(\alpha,\alpha+\frac{1}{m},\alpha+\frac{2}{m},\ldots,\alpha+\frac{m-1}{m}::\frac{ae^{i\pi}}{m^m t}) = \frac{1}{2\pi i}\int \Gamma(s)\prod_{\nu=0}^{m-1}\Gamma(\alpha+\frac{\nu}{m}-s)(\frac{ae^{i\pi}}{m^m t})^s\,ds$$

where the path of integration may be taken along the imaginary axis with loops, if necessary, to have the poles $\alpha+\nu/m$ lie to the right of the contour. If we consider the summation $\sum_{i,-i}\frac{1}{i}E$ called for in (25) we note that the factor $e^{i\pi s}$ will appear in the integral representation of the first term of the sum and the factor $e^{-i\pi s}$ will appear in the integral representation of the second term. Conversion of these factors to cosine and sine terms permits combination of the two integrals and the use of (8) gives the result that (25) equals AI where I is given

$$(27)\qquad I = -\frac{1}{2\pi i}\int \prod_{\nu=0}^{m-1}\frac{\Gamma(\alpha+\frac{\nu}{m}-s)}{\Gamma(1-s)}\left(\frac{a}{m^m t}\right)^s ds$$

with the same path of integration as before and where

$$A = (2\pi)^{\frac{1}{2}-\frac{m}{2}}\, m^{\frac{1}{2}+m\alpha}\, a^{-\alpha}.$$

We show next that

$$(28)\qquad \exp\left\{(m-1)\left(\frac{a}{m^m t}\right)^{\frac{1}{m-1}}\right\}(2\pi)^{1-\frac{m}{2}}(m-1)^{1/2}\, I = -\frac{1}{2\pi i}\int S(s)\left(\frac{a}{m^m t}\right)^s ds$$

where

$$S(s) = \sum_{u=0}^{\infty}\frac{\prod_{\nu=0}^{m-1}\Gamma(\alpha+\frac{u}{m-1}+\frac{\nu}{m}-s)}{\prod_{r=1}^{m-1}\Gamma(\frac{u+r}{m-1})\Gamma(1+\frac{u}{m-1}-s)}$$

when $R(s) > R(\alpha-\frac{1}{2})$. To see this we first substitute the value of S(s) in (28) and the right hand side becomes

$$(29)\qquad -\frac{1}{2\pi i}\int \sum_{u=o}^{\infty} \frac{\prod_{\nu=o}^{m-1}\Gamma(\alpha+\frac{u}{m-1}+\frac{\nu}{m}-s)}{\prod_{r=1}^{m-1}\Gamma(\frac{u+r}{m-1})\Gamma(1+\frac{u}{m-1}-s)}\left(\frac{a}{m^m t}\right)^s ds \,.$$

We interchange integration and summation. Then (29) becomes

$$(30)\qquad \sum_{u=o}^{\infty} -\frac{1}{2\pi i}\int \frac{\prod_{\nu=o}^{m-1}\Gamma(\alpha+\frac{u}{m-1}+\frac{\nu}{m}-s)}{\prod_{r=1}^{m-1}\Gamma(\frac{u+r}{m-1})\Gamma(1+\frac{u}{m-1}-s)}\left(\frac{a}{m^m t}\right)^s ds \,.$$

In (30) we change s to $s+\frac{u}{m-1}$ and obtain

$$\sum_{u=o}^{\infty} -\frac{1}{2\pi i}\left(\frac{a}{m^m t}\right)^{\frac{u}{m-1}} \frac{1}{\prod_{r=1}^{m-1}\Gamma(\frac{u+r}{m-1})}\int \frac{\prod_{\nu=o}^{m-1}\Gamma(\alpha+\frac{\nu}{m}-s)}{\Gamma(1-s)}\left(\frac{a}{m^m t}\right)^s ds.$$

This equals

$$\sum_{u=o}^{\infty}\left(\frac{a}{m^m t}\right)^{\frac{u}{m-1}} \frac{1}{\Gamma(\frac{u+1}{m-1})\Gamma(\frac{u+2}{m-1})\cdots\Gamma(\frac{u+1}{m-1}+\frac{m-2}{m-1})}\quad I \,.$$

We now use (7) with m-1 instead of m to obtain

$$\sum_{u=o}^{\infty}\left[\left(\frac{a}{m^m t}\right)^{\frac{1}{m-1}}\right]^u \frac{(2\pi)^{-\frac{m}{2}+1}(m-1)^{\frac{1}{2}+u}}{\Gamma(u+1)}\qquad I,$$

which equals

$$(2\pi)^{1-\frac{m}{2}}(m-1)^{1/2}\sum_{u=o}^{\infty}\left\{(m-1)\left(\frac{a}{m^m t}\right)^{\frac{1}{m-1}}\right\}^u \frac{1}{\underline{|u}}\quad I$$

or

$$(2\pi)^{1-\frac{m}{2}}(m-1)^{1/2}\exp\left[(m-1)\left(\frac{a}{m^m t}\right)^{\frac{1}{m-1}}\right]\quad I \,.$$

With (28) established, we utilize a known asymptotic expansion of the expression on the right side of (28), which holds when $\left|\frac{1}{t}\right|$ is large

(see [4], p.296-7). By putting all the factors on the left side of (28), except I on the right, we conclude that for $|t|$ near 0, or when $\frac{1}{|t|}$ is large,

$$AI \sim A \exp\left[-(m-1)\left(\frac{a}{m^m t}\right)^{\frac{1}{m-1}}\right]\left[\frac{a}{m^m t}\right]^{\frac{1}{m-1}(m\alpha-\frac{1}{2})}$$

(31)
$$\left\{\frac{(2\pi)^{\frac{m}{2}-1}}{\sqrt{m-1}} + \frac{M_1}{\left(\frac{a}{m^m t}\right)^{\frac{1}{m-1}}} + \frac{M_2}{\left(\frac{a}{m^m t}\right)^{\frac{2}{m-1}}} + \cdots\right\},$$

wherein the coefficients M_1, M_2, ... do not depend on t. Of course the leading term in this asymptotic expansion, in view of the value of A, is

$$(2\pi)^{-\frac{1}{2}} \frac{m^{\frac{1}{2}+m\alpha}}{\sqrt{m-1}} a^{-\alpha} \exp\left[-(m-1)\left(\frac{a}{m^m t}\right)^{\frac{1}{m-1}}\right]\left[\frac{a}{m^m t}\right]^{\frac{1}{m-1}(m\alpha-\frac{1}{2})} .$$

We now consider the special case of $m = 2$ and show that the result (31) agrees for this case with a known result. We suppose that $|\frac{a}{4t}|$ is large which, of course, will be the case for t near 0. We substitute $m = 2$ in (31) and taking account of the value of A obtain

(32)
$$e^{-a/4t}\left(\frac{a}{4t}\right)^{2\alpha-\frac{1}{2}}(2\pi)^{-\frac{1}{2}}\,2^{\frac{1}{2}+2\alpha}\,a^{-\alpha}\,J\left(\frac{a}{4t}\right),$$

where $J(\frac{a}{4t})$ tends to 1 as $|\frac{a}{4t}|$ tends to infinity.

This result agrees with the known asymptotic expansion of the original function (3) which holds for the case $m = 2$, because (see [2], p. 392, line 2)

$$D_{2\alpha-1}\left(\sqrt{\frac{a}{2t}}\right) = \frac{2^{-2\alpha}\left(\frac{a}{2t}\right)^{\alpha-\frac{1}{2}}}{\pi^{1/2}\Gamma(1-2\alpha)} e^{-a/8t} E\left(\frac{1}{2}-\alpha, 1-\alpha :: \frac{a}{4t}\right) .$$

However (see [3], p. 351, formula (13)),

$$E\left(\frac{1}{2}-\alpha, 1-\alpha :: \frac{a}{4t}\right) \sim \Gamma\left(\frac{1}{2}-\alpha\right)\Gamma(1-\alpha)\, J\left(\frac{a}{4t}\right)$$

where $J(\frac{a}{4t}) \longrightarrow 1$ as $t \longrightarrow 0$. Hence

$$D_{2\alpha-1}\left(\sqrt{\frac{a}{2t}}\right) \sim \frac{2^{-2\alpha}\left(\frac{a}{2t}\right)^{\alpha-\frac{1}{2}}}{\pi^{1/2}\Gamma(1-2\alpha)} e^{-a/8t}\, \Gamma(\tfrac{1}{2}-\alpha)\Gamma(1-\alpha)J(\tfrac{a}{4t}) .$$

Use of the Legendre duplication formula for the gamma function gives

$$D_{2\alpha-1}\left(\sqrt{\frac{a}{2t}}\right) \sim \left(\frac{a}{2t}\right)^{\alpha-\frac{1}{2}} e^{-a/8t}\, J(\tfrac{a}{4t}) .$$

Hence the asymptotic value of the original function (3) is

$$2^{\frac{1}{2}-\alpha}\, \pi^{-1/2}\, t^{-\alpha} \left(\frac{a}{2t}\right)^{\alpha-\frac{1}{2}} e^{-a/4t}\, J(\tfrac{a}{4t})$$

which agrees with (32).

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Appendix

We now prove a relationship used in the text proper, namely,

$$(1)\qquad \int_0^{\infty} e^{-\mu}\,\mu^{\alpha_{p+1}-1} E(p;\alpha_r:q;\rho_s:z/\mu)\,d\mu = E(p+1;\alpha_r:q;\rho_s:z)$$

when $R(\alpha_{p+1}) > 0$. (This relationship is stated as problem 106 on p. 394 of [3].)

Proof: If $p \geq q+1$, we use the definition of the E-function ([3], p. 352, eq. (20)), namely,

$$(2)\qquad \begin{aligned} E(p;\alpha_r:q;\rho_s:z) = \Gamma(\alpha_p) \left\{ \prod_{n=1}^{q} \Gamma(\rho_n-\alpha_n) \right\}^{-1} \prod_{n=1}^{q} \int_0^1 \lambda_n^{\alpha_n-1} (1-\lambda_n)^{\rho_n-\alpha_n-1} d\lambda_n \cdot \\ \prod_{n=q+1}^{p-2} \int_0^{\infty} e^{-\lambda_n} \lambda_n^{\alpha_n-1} d\lambda_n \int_0^{\infty} e^{-\lambda_{p-1}} \lambda_{p-1}^{\alpha_{p-1}-1} (1+\lambda_1\lambda_2\ldots\lambda_{p-1}/z)^{-\alpha_p} d\lambda_{p-1} \end{aligned}$$

where $R(\alpha_n) > 0$, $n = 1,2,\ldots,p-1$, $R(\rho_n-\alpha_n) > 0$, $n = 1,2,\ldots,q$. We substitute this expression for E given by (2) with of course z replaced by z/μ into (1). We then merely place the terms involving μ and the integration with respect to μ within the innermost integral. The resulting expression is seen to be, by the very definition of the E-function,

$$(3)\qquad E(\alpha_1,\alpha_2,\ldots,\alpha_{p-1},\ \alpha_{p+1},\ \alpha_p:q;\rho_s:z)\ .$$

However, equation (5) of the text shows that the E-function is symmetric in the α_r and ρ_s. Hence expression (3) is equal to the right side of (1).

If $p \leq q$, we use as the value of the E-function the definition for this case ([3], p. 352, eq. (21)), namely,

$$E(p;\alpha_r:q;\rho_s:z) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\ldots\Gamma(\alpha_p)}{\Gamma(\rho_1)\Gamma(\rho_2)\ldots\Gamma(\rho_q)} F(p;\alpha_r:q;\rho_s:\ -\tfrac{1}{z})\ .$$

We substitute this value of E in (1), expand the generalized hypergeometric function into the series which defines it and integrate term by term. In view of the definition of the Gamma function, the result follows at once.

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